# Check polynomials and parity check matrices for cyclic codes

Let $C$ be a cyclic $[n,k]$-code with the generator polynomial $g(x)$ (of degree $n - k$). By the last theorem $g(x)$ is a factor of $x^n - 1$. Hence

$$x^n - 1 = g(x)h(x)$$

for some $h(x)$ of degree $k$ (where $h(x)$ is called the check polynomial of $C$).

Theorem Let $C$ be a cyclic code in $R_n$ with a generator polynomial $g(x)$ and a check polynomial $h(x)$. Then an $c(x) \in R_n$ is a codeword of $C$ if $c(x)h(x) \equiv 0$ - this and next congruences are modulo $x^n - 1$.

Proof Note, that $g(x)h(x) = x^n - 1 \equiv 0$

(i) $c(x) \in C \Rightarrow c(x) = a(x)g(x)$ for some $a(x) \in R_n$

$$\Rightarrow c(x)h(x) = a(x)\underbrace{g(x)h(x)}_{\equiv 0} \equiv 0.$$

(ii) $c(x)h(x) \equiv 0$

$$c(x) = q(x)g(x) + r(x),\ deg\ r(x) < n - k = deg\ g(x)$$

$$c(x)h(x) \equiv 0 \Rightarrow r(x)h(x) \equiv 0 \pmod{x^n - 1}$$

Since $deg\ (r(x)h(x)) < n - k + k = n$, we have $r(x)h(x) = 0$ in $F[x]$ and therefore

$$r(x) = 0 \Rightarrow c(x) = q(x)g(x) \in C.$$

# POLYNOMIAL REPRESENTATION of DUAL CODES

Since $dim(\langle h(x)\rangle) = n - k = dim(C^{\perp})$ we might easily be fooled to think that the check polynomial $h(x)$ of the code $C$ generates the dual code $C^{\perp}$.

Reality is “slightly different":

Theorem Suppose $C$ is a cyclic $[n,k]$-code with the check polynomial

$$h(x) = h_0 + h_1x + \dots + h_kx^k,$$

then

(i) a parity-check matrix for $C$ is

$$H = \begin{pmatrix} h_k & h_{k-1} & \dots & h_0 & 0 & \dots & 0 \\ 0 & h_k & \dots & h_1 & h_0 & \dots & 0 \\ .. & .. & & & & & \\ 0 & 0 & \dots & 0 & h_k & \dots & h_0 \end{pmatrix}$$

(ii) $C^{\perp}$ is the cyclic code generated by the polynomial

$$\bar{h}(x) = h_k + h_{k-1}x + \dots + h_0x^k$$

i.e. the reciprocal polynomial of $h(x)$.

# POLYNOMIAL REPRESENTATION of DUAL CODES

Proof A polynomial $c(x) = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1}$ represents a code from $C$ if $c(x)h(x) = 0$. For $c(x)h(x)$ to be 0 the coefficients at $x^k, \ldots, x^{n-1}$ must be zero, i.e.

$$\begin{aligned} c_0 h_k + c_1 h_{k-1} + \ldots + c_k h_0 &= 0 \\ c_1 h_k + c_2 h_{k-1} + \ldots + c_{k+1} h_0 &= 0 \\ .. \qquad\qquad & .. \\ c_{n-k-1} h_k + c_{n-k} h_{k-1} + \ldots + c_{n-1} h_0 &= 0 \end{aligned}$$

Therefore, any codeword $c_0 c_1 \ldots c_{n-1} \in C$ is orthogonal to the word $h_k h_{k-1} \ldots h_0 00 \ldots 0$ and to its cyclic shifts.

Rows of the matrix $H$ are therefore in $C^\perp$. Moreover, since $h_k = 1$, these row-vectors are linearly independent. Their number is $n - k = dim\ (C^\perp)$. Hence $H$ is a generator matrix for $C^\perp$, i.e. a parity-check matrix for $C$.

In order to show that $C^\perp$ is a cyclic code generated by the polynomial

$$\bar{h}(x) = h_k + h_{k-1} x + \ldots + h_0 x^k$$

it is sufficient to show that $\bar{h}(x)$ is a factor of $x^n - 1$.

Observe that $\bar{h}(x) = x^k h(x^{-1})$ and since $h(x^{-1})g(x^{-1}) = (x^{-1})^n - 1$

we have that $x^k h(x^{-1}) x^{n-k} g(x^{-1}) = x^n (x^{-n} - 1) = 1 - x^n$

and therefore $\bar{h}(x)$ is indeed a factor of $x^n - 1$.

# ENCODING with CYCLIC CODES I

Encoding using a cyclic code can be done by a multiplication of two polynomials - a message polynomial and the generating polynomial for the cyclic code.

Let $C$ be an $(n,k)$-code over an field $F$ with the generator polynomial
$g(x) = g_0 + g_1 x + \ldots + g_{r-1} x^{r-1}$ of degree $r = n - k$.

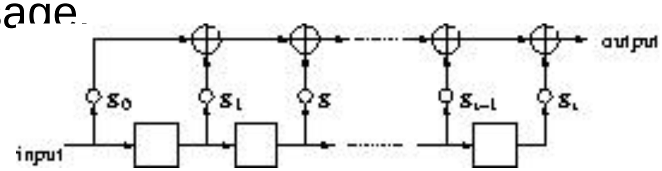
If a message vector $m$ is represented by a polynomial $m(x)$ of degree $k$ and $m$ is encoded by

$$m \Rightarrow c = mG_1,$$

then the following relation between $m(x)$ and $c(x)$ holds

$$c(x) = m(x)g(x).$$

Such an encoding can be realized by the shift register shown in Figure below, where input is the $k$-bit message to be encoded followed by $n - k$ 0' and the output will be the encoded message.




Shift-register encodings of cyclic codes. Small circles represent multiplication by the corresponding constant, ⊕ nodes represent modular addition, squares are delay elements

# ENCODING of CYCLIC CODES II

Another method for encoding of cyclic codes is based on the following (so called systematic) representation of the generator and parity-check matrices for cyclic codes.

Theorem Let $C$ be an $(n,k)$-code with generator polynomial $g(x)$ and $r = n - k$. For $i = 0,1,\ldots,k - 1$, let $G_{2,i}$ be the length $n$ vector whose polynomial is $G_{2,i}(x) = x^{r+I} - x^{r+I}$ mod $g(x)$. Then the $k * n$ matrix $G_2$ with row vectors $G_{2,I}$ is a generator matrix for $C$.

Moreover, if $H_{2,J}$ is the length $n$ vector corresponding to polynomial $H_{2,J}(x) = x^j$ mod $g(x)$, then the $r * n$ matrix $H_2$ with row vectors $H_{2,J}$ is a parity check matrix for $C$. If the message vector $m$ is encoded by

$$m \Rightarrow c = mG_2,$$

then the relation between corresponding polynomials is

$$c(x) = x^r m(x) - [x^r m(x)] \bmod g(x).$$

On this basis one can construct the following shift-register encoder for the case of a systematic representation of the generator for a cyclic code:

Shift-register encoder for systematic representation of cyclic codes. Switch A is closed for first $k$ ticks and closed for last $r$ ticks; switch B is down for first $k$ ticks and up for last $r$ ticks.

# Hamming codes as cyclic codes

Definition (Again!) Let $r$ be a positive integer and let $H$ be an $r$ * ($2^r$ -1) matrix whose columns are distinct non-zero vectors of $V(r,2)$. Then the code having $H$ as its parity-check matrix is called binary **Hamming code** denoted by *Ham* $(r,2)$.

It can be shown that binary Hamming codes are equivalent to cyclic codes.

Theorem The binary Hamming code *Ham* $(r,2)$ is equivalent to a cyclic code.

Definition If p($x$) is an irreducible polynomial of degree $r$ such that $x$ is a primitive element of the field $F[x]$ / p($x$), then p($x$) is called a primitive polynomial.

Theorem If p($x$) is a primitive polynomial over $GF(2)$ of degree $r$, then the cyclic code $\langle p(x) \rangle$ is the code *Ham* $(r,2)$.

# Hamming codes as cyclic codes

**Example** Polynomial $x^3 + x + 1$ is irreducible over $GF(2)$ and $x$ is primitive element of the field $F_2[x] / (x^3 + x + 1)$.

$$F_2[x] / (x^3 + x + 1) =$$

$$\{0, x, x^2, x^3 = x + 1, x^4 = x^2 + x, x^5 = x^2 + x + 1, x^6 = x^2 + 1\}$$

The parity-check matrix for a cyclic version of *Ham* $(3,2)$

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

# PROOF of THEOREM

The binary Hamming code *Ham* $(r,2)$ is equivalent to a cyclic code.

It is known from algebra that if $p(x)$ is an irreducible polynomial of degree $r$, then the ring $F_2[x] / p(x)$ is a field of order $2^r$.

In addition, every finite field has a primitive element. Therefore, there exists an element $\alpha$ of $F_2[x] / p(x)$ such that

$$F_2[x] / p(x) = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2r-2}\}.$$

Let us identify an element $a_0 + a_1 + \ldots a_{r-1}x^{r-1}$ of $F_2[x] / p(x)$ with the column vector

$$(a_0, a_1, \ldots, a_{r-1})^T$$

and consider the binary $r * (2^r - 1)$ matrix

$$H = [\,1 \;\; \alpha \;\; \alpha^2 \; \ldots \; \alpha^{2^\wedge r-2}\,].$$

Let now $C$ be the binary linear code having $H$ as a parity check matrix.

Since the columns of $H$ are all distinct non-zero vectors of $V(r,2)$, $C = Ham\ (r,2)$.

Putting $n = 2^r - 1$ we get

$$C = \{f_0 f_1 \ldots f_{n-1} \in V(n, 2) \mid f_0 + f_1 \alpha + \ldots + f_{n-1} \alpha^{n-1} = 0 \qquad (2)$$

$$= \{f(x) \in R_n \mid f(\alpha) = 0 \text{ in } F_2[x] / p(x)\} \qquad (3)$$

If $f(x) \in C$ and $r(x) \in R_n$, then $r(x)f(x) \in C$ because

$$r(\alpha)f(\alpha) = r(\alpha) \bullet 0 = 0$$

and therefore, by one of the previous theorems, this version of *Ham* $(r,2)$ is cyclic.

# BCH codes and Reed-Solomon codes

To the most important cyclic codes for applications belong BCH codes and Reed-Solomon codes.

Definition A polynomial $p$ is said to be *minimal* for a complex number $x$ in $Z_q$ if $p(x) = 0$ and p is irreducible over $Z_q$.

Definition A cyclic code of codewords of length $n$ over $Z_q$, $q = p^r$, $p$ is a prime, is called BCH code[1] of distance $d$ if its generator $g(x)$ is the least common multiple of the minimal polynomials for

$$\omega^{l}, \omega^{l+1}, \ldots, \omega^{l+d-2}$$

for some l, where

$\omega$ is the primitive $n$-th root of unity.

If $n = q^m - 1$ for some $m$, then the BCH code is called primitive.

Definition A Reed-Solomon code is a primitive BCH code with $n = q - 1$.

Properties:

- Reed-Solomon codes are self-dual.

[1]BHC stands for Bose and Ray-Chaudhuri and Hocquenghem who discovered these codes.

# CONVOLUTION CODES

**Very often it is important to encode an infinite stream or several streams of data – say bits.**

**Convolution codes, with simple encoding and decoding, are quite a simple generalization of linear codes and have encodings as cyclic codes.**

**An *(n,k)* convolution code (CC) is defined by an *k x n generator matrix, entries of which are polynomials over* $F_2$**

**For example,**

$$G_1 = [x^2 + 1, x^2 + x + 1]$$

**is the generator matrix for a (2,1) convolution code $CC_1$ and**

$$G_2 = \begin{pmatrix} 1+x & 0 & x+1 \\ 0 & 1 & x \end{pmatrix}$$

**is the generator matrix for a (3,2) convolution code $CC_2$**

# ENCODING of FINITE POLYNOMIALS

An (n,k) convolution code with a k x n generator matrix G can be usd to encode a k-tuple of plain-polynomials (polynomial input information)

$$I=(I_0(x), I_1(X),\ldots,I_{k-1}(x))$$

to get an n-tuple of crypto-polynomials

$$C=(C_0(x), C_1(x),\ldots,C_{n-1}(x))$$

As follows

$$C= I \cdot G$$

# EXAMPLES

## EXAMPLE 1

$$(x^3 + x + 1).G_1 = (x^3 + x + 1) \,.\, (x^2 + 1,\ x^2 + x + 1]$$
$$= (x^5 + x^2 + x + 1,\ x^5 + x^4 + 1)$$

## EXAMPLE 2

$$(x^2 + x, x^3 + 1).G_2 = (x^2 + x, x^3 + 1).\begin{pmatrix} 1\,0\,x+1 \\ 0\,1\quad x \end{pmatrix}$$

# ENCODING of INFINITE INPUT STREAMS

**The way infinite streams are encoded using convolution codes will be Illustrated on the code $CC_1$.**

**An input stream $I = (I_0, I_1, I_2, \ldots)$ is mapped into the output stream $C = (C_{00}, C_{10}, C_{01}, C_{11} \ldots)$ defined by**

$$C_0(x) = C_{00} + C_{01}x + \ldots = (x^2 + 1)\, I(x)$$

**and**

$$C_1(x) = C_{10} + C_{11}x + \ldots = (x^2 + x + 1)\, I(x).$$

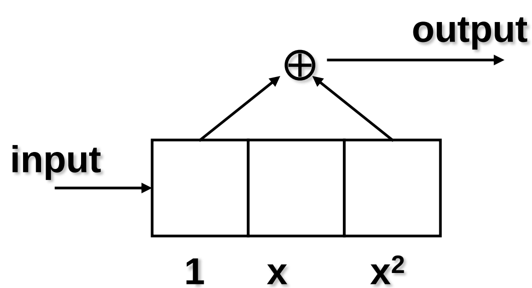
**The first multiplication can be done by the first shift register from the next figure; second multiplication can be performed by the second shift register on the next slide and it holds**

$$C_{0i} = I_i + I_{i+2}, \qquad C_{1i} = I_i + I_{i-1} + I_{i-2}.$$

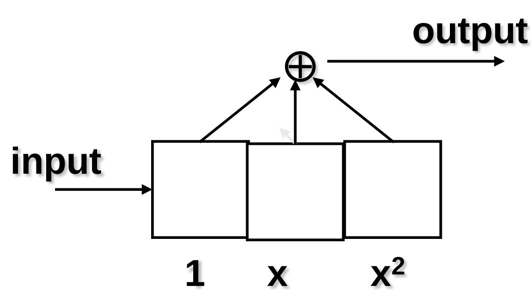
***That is the output streams $C_0$ and $C_1$ are obtained by convolving the input stream with polynomials of $G_1$.***

# ENCODING

The first shift register

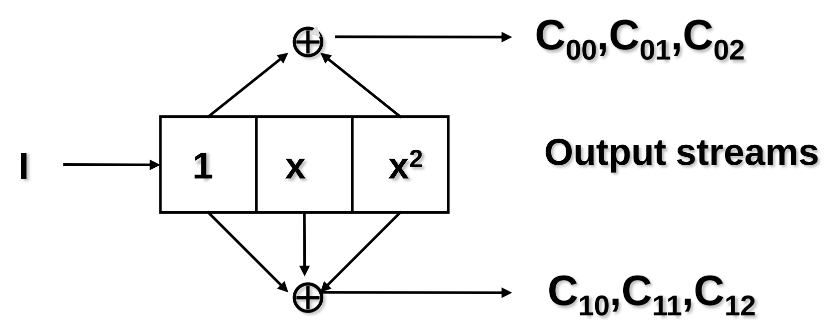



**will multiply the input stream by $x^2$+1 and the second shift register**




**will multiply the input stream by $x^2+x+1$.**

**The following shift-register will therefore be an encoder for the code $CC_1$**




**For encoding of convolution codes so called**

**Viterbi algorithm**

**Is used.**